

The equilibrium state of 2D foams

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Abstract. – The dynamics of two-dimensional cellular networks (foams) is written in terms of coupled rate equations, which describe how the population of s -sided cells is affected by cell disappearance or coalescence and division. In these equations, the effect of the rest of the foam in statistical equilibrium on the disappearing or dividing cell is treated as a local mean field. The rate equations are asymptotically integrable; the equilibrium distribution P_s of cells is essentially unique, driven and controlled by the topological transformations for cells with $s < 6 + \sqrt{\mu_2}$. Asymptotic integrability of the equations, and unique distribution, are absent in a global mean-field treatment. Thus, short-ranged topological information is necessary to explain the evolution and stability of foams.

Introduction. – Two-dimensional random cellular networks (“2D foams”) are widespread in nature (soap froths, fragmentation patterns, biological epidermis, etc.) [1]. They are random partitions of the plane by cells, which are topological polygons [1–3]. Disorder or absence of specific adjustment imposes minimal incidence numbers (3 edges incident on a vertex). Foams evolve into a stationary state of statistical equilibrium, with an invariant distribution of cell shapes P_s , where s , the number of sides of a cell, is the only topological random variable [1,4]. Statistical equilibrium is established through local, elementary topological transformations (ETT), which can be an edge flip (T1 transformation) or the disappearance of a 3-sided cell (T2 transformation). In biological tissues, combinations of these transformations constitute cell division (m , for mitosis) or its inverse, cell disappearance (d) (coalescence of two cells by removal of their interface). It turns out that the asymptotic behaviour of the distribution P_s is “universal”. We show here that this universality is due to asymptotic integrability of the equations describing the variations of the cell population P_s under ETT.

Rate equations in the local mean field approximation. – The stationary distribution P_s for a foam is the solution of coupled rate equations, which account for the local, but correlated

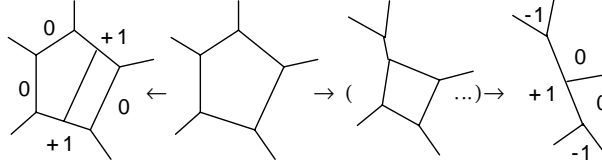


Fig. 1 – Topological transformations: division (m) and disappearance (d) of a 5-sided cell. The numbers $0, \pm 1$ are the numbers of sides gained by the neighbouring cells in the process.

variations in the population of cells under ETT [5–7]:

$$\sum_k P_k P(m|k) [-\delta_{ks} + \Gamma(s|k; m) + (2/k)M_{s-1}(k) - (2/k)M_s(k) - P_s] + \sum_k P_k P(d|k) \left[-\delta_{ks} - M_s(k) + \sum_{i=-1}^{k-4} M_{s-i}(k) \Delta(i|k; d) + P_s \right] = 0. \quad (1)$$

In (1), the first square bracket deals with cell division, the second with cell disappearance⁽¹⁾. The rate equations depend on (see, *e.g.*, fig. 1):

- $P(m|k)$ ($P(d|k)$): the conditional probability that an existing k -cell divides (disappears), weighted by the rate of division (disappearance).
- $\Gamma(s|k; m)$: the conditional probability that a k -sided dividing cell has an s -sided daughter ($3 \leq s \leq k + 1$).
- $\Delta(i|k; d)$: the conditional probability that a k -sided disappearing cell gives i sides to one of its neighbours ($-1 \leq i \leq k - 4$).
- $M_s(k) = A_{s,k} P_s$: the average number of s -cell neighbours of a k -cell.

A geometrical solution must satisfy the constraints

$$\begin{aligned} \sum_s P_s &= 1 \quad (\text{normalization}), \\ \sum_s P_s &= 6 \quad (\text{from Euler's relation for a polygonal foam}), \\ \sum_s M_s(k) &= \sum_s A_{s,k} P_s = k \quad (\text{a } k\text{-sided cell has } k \text{ neighbours}). \end{aligned} \quad (2)$$

⁽¹⁾The last term in each bracket, $\pm P_s$, expresses the disappearance/production of one cell during the topological transformation (d)/(m).

The constraints are identically satisfied by P_s if the parameters obey the sum rules [5, 7]⁽²⁾

$$\begin{aligned}
\sum_s \Gamma(s|k; m) &= 2, \\
\sum_s s\Gamma(s|k; m) &= k + 4, \\
\sum_{i=-1}^{k-4} \Delta(i|k; d) &= 1, \\
\sum_{i=-1}^{k-4} i\Delta(i|k; d) &= 1 - (6/k), \\
\Delta(-1|k; d) &= 1/3 + 4(k-3)!/[k!].
\end{aligned} \tag{3}$$

The third constraint sets the local mean-field approximation: Maximum entropy infers that, in statistical equilibrium, the correlator $A_{k,s}$ is linear in k and s [8], $A_{k,s} = (k-6)\sigma(s-6) + (s+k-6)$, where σ is a structural parameter [6, 8]. The third constraint is then redundant, automatically satisfied by the other two. It is through the s dependence of the parameter $M_s(k)$ that the local environment of the cell affected by ETT is represented.

The system of integro-difference equations (1) is greatly simplified if either the cell division is the only topological process to take place ($P(d|k) = 0$), with $\Gamma(s|k; m)$ independent of s for $3 \leq s \leq k+1$ [6]⁽³⁾, or if only three-, four- and five-sided cells can disappear ($P(d|k) = 0$ for $k \geq 6$) and there is no cell division ($P(m|k) = 0$). The rate equations (1) for P_s are then a second-order linear difference equation:

$$a(s+1)P_{s+1} + b(s)P_s + c(s-1)P_{s-1} = 0, \tag{4}$$

for $s \geq 6$ in the case of cell disappearance, and for all $s \geq 3$ ($P_{s \leq 2} = 0$) in the case of cell division. The coefficients $a(s)$, $b(s)$ and $c(s)$ of (4) depend on s , but they are all polynomials of the *same* degree n , $a(s) = A_n s^n + A_{n-1} s^{n-1} + \dots + A_0$, etc. This degree is 2 for the division/fragmentation equations [6]. The degree of the polynomials $a(s)$, $b(s)$ and $c(s)$ is 1 for the cell death/coalescence equations (see eq. (8) below), in the local mean-field approximation. Examples are given in [6], in table I, and below.

Asymptotic integrability of the rate equations. – The constraints (2) make eq. (4) asymptotically integrable (integrable for large s). Asymptotic integrability selects the physical solution of the second-order difference equation and filters out the irrelevant one.

If $a(s) + b(s) + c(s) = 0$, eq. (4) reads $D[a(s+1)P_{s+1} - c(s)P_s] = 0$, where $Df(s) = f(s) - f(s-1)$. The second-order equation (4) is *integrable* and becomes first-order:

$$a(s+1)P_{s+1} - c(s)P_s = 0, \tag{5}$$

⁽²⁾The relations for Γ express the fact that a k -sided cell divides into two daughter cells with s and $k+4-s$ sides. The relations for Δ are obtained [7] by assuming that the k -sided cell disappears through successive disappearance of its sides (fig. 1), starting with the smallest one. For $k=3$, $\Delta(-1|3; d) = 1$. This yields the recursion relations in k and i :

$k\Delta(-1|k; d) = (k-3)\Delta(-1|k-1; d) + 1$, for $k \geq 4$,

$k\Delta(i|k; d) = (k-3)\Delta(i|k-1; d) + 2\Delta(i-1|k-1; d)$, for $0 \leq i \leq k-4$.

⁽³⁾Subtracting the equation for s from the equation for $s+1$ eliminates the “integral” $\sum_k P_k \Gamma(s|k; m)$.

TABLE I – *Asymptotic integrability of the rate equations in the steady-state, and the distribution $P_s \sim Cs^{-\kappa}z^s$, for various natural and simulated foams.*

Local topological transfer	Integrability	n	z	κ
<i>Division</i>				
S_1 [6]	$q(s) = 0$, integr.	2	1	$1/A_2$
S_2 [6]	$q(s) = s - 6$, $n_q = 1$	2	$2/3$	0
			1	3
<i>Disappearance</i>				
Local mean field	$q(s) = 1$, $n_q = 0$	1	C_1/A_1	$1 - 1/(A_1 - C_1) +$ $+A_0/A_1 - C_0/C_1$
			1	$1 + 1/(A_1 - C_1)$
Topological gas [4, 6, 9]	$q(s) = 1$, $A_0 = 0 = C_0$	1	C_1/A_1	$1 - 1/(A_1 - C_1)$
$A_{ks} = ks/6$, $\sigma = 1/6$			1	$1 + 1/(A_1 - C_1)$
<i>Random T1</i>				
Simulations [10]	–	–	$3/4$?
Simulations [3]	–	–	0.74	?

with the constant = 0 on the right-hand side, because the solution P_s must satisfy the constraints (2). It has the unique solution

$$P_s = [c(s-1)!/a(s)!] \{ [a(i)!/c(i-1)!] P_i \} = P_s^{(1)}, \quad (6)$$

with $a(s)! = a(s)a(s-1)a(s-2) \cdots a(4)a(3)$, and a boundary value P_i (for $i = 5$, say). Asymptotically, $P_s^{(1)} \sim Cs^{-\kappa}z^s$ decays exponentially, with $z = C_n/A_n$ and $\kappa = n + (A_{n-1}/A_n - C_{n-1}/C_n)$.

Equation (4) is *asymptotically integrable* (integrable at infinity) because

$$a(s) + b(s) + c(s) = q(s), \quad (7)$$

where $q(s)$ is a polynomial of lower degree $n_q < n$. Equation (4) can then be written as $D[a(s+1)P_{s+1} - c(s)P_s] + q(s)P_s = 0$. Since $n_q < n$, it reduces, for large s , to the first-order equation (5).

The linear equation (4) has two independent solutions of the type $P_s \sim Cs^{-\kappa}z^s$ [11]. The first solution $P_s^{(1)}$ is the same as that of the first-order equation (5). It decays exponentially, with $z = C_n/A_n < 1$, and $\kappa = n + [C_n(A_{n-1}/A_n) + B_{n-1} + A_n(C_{n-1}/C_n)]/(C_n - A_n)$. The second solution $P_s^{(2)}$ is algebraic, $z = 1$ and $\kappa = n - [C_{n-1} + B_{n-1} + A_{n-1}]/(C_n - A_n)$.

The general solution of (4) is $P_s = F_1P_s^{(1)} + F_2P_s^{(2)}$, but F_2 is negligible: All foams generated by local elementary topological transformations (ETT) are described by the exponentially decaying solution $P_s^{(1)}$. This is because the physical process of statistical equilibrium under ETT, constrained by $\langle s \rangle = 6$, has nearly exhausted all the nonvanishing $P_s^{(1)}$ by $s \approx 6 + \sqrt{\mu_2}$. The algebraic solution $P_s^{(2)}$ is a negligible, smooth background, with $F_2 [\zeta(\kappa) - 1 - 2^{-\kappa}] \ll 1$. (ζ is Riemann's zeta function).

Example: cell division only [6]. – i) Algorithm S_2 (random selection of an edge of the dividing cell), $P(m|k) = k/6$. The algebraic tail ($\kappa = 3$) appears for $s > 125$, where $P_{125}^{(1)} = 10^{-21} = F_2(125)^{-3}$, thus $F_2 = 10^{-14}$. $\mu_2 = 8.17$. ii) By contrast, the algorithm S_1 of [6] (random selection of the dividing cell), $P(m|k) = 1$, yields an integrable rate equation, with $n = 2$, $C_2 = A_2 = [\langle m(s) \rangle - 6]/\mu_2 > 0$ ($m(s)$ is the average number of sides of cells neighbours to an s -sided cell). $z = 1$, the decay is algebraic, $P_s \sim s^{-\kappa}$ with $\kappa = 1/A_2$.

Example: cell disappearance. – One assumes that only three-, four- and five-sided cells can disappear, in agreement with von Neumann's law [1, 4] and with simulations [7]. Thus, $P(d|k) = 0$ for $k \geq 6$. The sum rules (3) yield $\Delta(-1|3; d) = 1$, $\Delta(-1|4; d) = \Delta(0|4; d) = 1/2$, $\Delta(-1|5; d) = \Delta(0|5; d) = 2\Delta(1|5; d) = 2/5$. The polynomials

$$\begin{aligned} a(s) &= \gamma(3)A_{s,3} + [\gamma(4)/2]A_{s,4} + [2\gamma(5)/5]A_{s,5}, \\ b(s) &= -\gamma(3)A_{s,3} - [\gamma(4)/2]A_{s,4} - [3\gamma(5)/5]A_{s,5} + 1, \\ c(s) &= [\gamma(5)/5]A_{s,5} \end{aligned} \quad (8)$$

are then linear in s in a local mean-field approximation, for $s \geq 6$, where $\gamma(k) = P_k P(d|k) / \sum_s [P_s P(d|s)]$. The correlators $A_{s,k}$ have been defined in eq. (2). Thus,

$$a(s) + b(s) + c(s) = 1, \quad (9)$$

and eq. (4) has asymptotic integrability. As in the cell division case, the physical solution P_s is unique with $\langle s \rangle = 6$, $\mu_2 < 10$, and it decays exponentially for large s . It is only for extremely large values of s , where P_s is negligible, that the algebraic tail dominates the exponential.

In the global mean-field approximation [12], a , b and c are constants, still adding up to one, eq. (4) is no longer asymptotically integrable, and one must use *ad hoc* criteria to select the physical solution which is exponentially decaying. Flyvbjerg's [13] is also a global mean-field approximation; the edges of the disappearing cell are redistributed to any cell of the foam, but in proportion to the number of sides of the latter. The coefficients of (4) are polynomials of degree 1, summing up to $q(s) = 1$, and the rate equation is asymptotically integrable. But Flyvbjerg ascribed the universality of the distribution P_s not to the asymptotic integrability of the equation as argued here, but to rather artificial initial conditions involving the probabilities of 2-, 1- and even zero-sided cells. Flyvbjerg's global mean-field approximation is mathematically identical to the "topological gas" ($\sigma = 1/6$, no correlation between neighbours, $A_{ks} = sk/6$), which is only a limit, inaccessible experimentally or through simulations [4, 6, 9]. It is also the approximation used by Marder, Beenakker and Yekutieli (see [10] and references therein).

Conclusions. – Local, topological correlations between cells determine the stationary state of foams. In a local mean-field approximation which accounts for nearest-neighbour correlations in statistical equilibrium, the rate equations are asymptotically integrable and have one unique solution with exponential asymptotic behaviour in s . The asymptotic integrability of the equation, which imposes a unique and universal solution, is absent in the global mean-field approximation. Thus, local environment and statistical equilibrium mould the stationary distribution of cell shapes, as had been surmised in the epidermis of mammals [7], and confirmed in simulations [4]. The single exponential decay of the physical distribution $P_s \sim \exp[-\beta s]$, imposed by asymptotic integrability, is indeed the Boltzmann distribution resulting from the conservation of $\langle s \rangle$, through maximum entropy [1, 4].

The same asymptotic integrability holds for cell division and disappearance combined, but the difference equation is then of order three.

The foams discussed here are purely topological and combinatorial, without any control through the energy carried by interfaces. To impose constant energy, and control disorder, the variance μ_2 of the distribution P_s should also be constrained, besides (2). As a consequence, P_s would have a Gaussian tail for very large s , with an additional equation of state, Lemaître's law, relating μ_2 to $1 - P_6$ [14].

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REFERENCES

- [1] WEAIRE D. and RIVIER N., *Contemp. Phys.*, **25** (1984) 59.
- [2] DUBERTRET B., RIVIER N. and PESHKIN M. A., *J. Phys. A*, **31** (1998) 879.
- [3] DUBERTRET B., ASTE T., OHLENBUSCH H. M. and RIVIER N., *Phys. Rev. E*, **58** (1998) 6368; OHLENBUSCH H. M., RIVIER N., ASTE T. and DUBERTRET B., *DIMACS Ser. Discrete Math. Theor. Compu. Sci.*, **51** (2000) 279.
- [4] RIVIER N., *Disorder and Granular Media*, edited by D. BIDEAU and A. HANSEN (Elsevier) 1993, p. 55.
- [5] RIVIER N., ARCENEGUI-SIEMENS X. and SCHLIECKER G., *Fragmentation Physics*, edited by D. BEYSENS, X. CAMPI and E. PEFFERKORN (World Scientific) 1995, p. 267.
- [6] DELANNAY R. and LECAËR G., *Phys. Rev. Lett.*, **73** (1994) 1553.
- [7] DUBERTRET B. and RIVIER N., *Biophys. J.*, **73** (1997) 38; DUBERTRET B., *Etude théorique des systèmes cellulaires désordonnés. Application à la modélisation des tissus épithéliaux*, PhD Thesis, Strasbourg (1998).
- [8] PESHKIN M. A., STRANDBURG K. J. and RIVIER N., *Phys. Rev. Lett.*, **67** (1991) 1803.
- [9] SCHLIECKER G. and KLAPP S., *Europhys. Lett.*, **48** (1999) 122.
- [10] YEKUTIELI I., *Tigures de souffle et structures cellulaires: aspects géométriques de la croissance*, PhD Thesis, Paris 6 (1992).
- [11] BENDER C. M. and ORSZAG S. A., *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill) 1978.
- [12] STAVANS J., DOMANY E. and MUKAMEL D., *Europhys. Lett.*, **15** (1991) 479.
- [13] FLYVBJERG H., *Phys. Rev. E*, **47** (1993) 4037.
- [14] RIVIER N., *Foams and Emulsions*, edited by J. F. SADOE and N. RIVIER (Kluwer) 1999, p. 105.